Jet schemes of rational double point singularities

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Abstract

We prove that for $m \in \mathbb{N},\ m$ big enough, the number of irreducble components of the schemes of $m-$jets centered at a point which is a double point singularity is independent of m and is equal to the number of exceptional curves on the minimal resolution of the singularity. We also relate some irreducible components of the jet schemes of an E_6 singularity to its "minimal" embedded resolutions of singularities.

1 Introduction

In this note, we are interested in schemes of jets centered at a rational double point singularity. While we have that the global jet schemes of surfaces having such a singularity is irreducible [Mu], we prove that for $m \in \mathbb{N},\ m$ big enough, the number $N(m)$ of irreducble components of the schemes of $m-$jets centered at the singular point is independent of m. Note that this is not the case for instance for plane curves [Mo1]. This may not be the case also when we have rational singularities which are not locally complete intersection [Mo2]. Moreover, we find that $N(m)$ is equal to the number of exceptional curves on the minimal resolution of the singularity. This reminds of the Nash map, which defines a correspondence between the irreducible components of the space of arcs centered at the singularity and the exceptional curves on the minimal resolution of the singularity. But in general, there is no direct relation between the irreducbile components of jet schemes and those of the arc space. For instance, the function $N(m)$, number of irreducible components of the schemes of $m-$jets centered in the singular point of a plane curve, goes to infinity when m goes to infinity [Mo1], while the space of arcs centered at the singular point is irreducible. Note that the Nash map is bijective for rational double point singularities [P1],[PS],[Pe],[Le], for surfaces with rational singularities [Re1],[Re2] and in general for surface singularities [deBPe].
We will also give a special treatment to the jet schemes of the E_6 singularity, by defining a bijective correspndence between some irreducible components of some jet schemes and the divisors appearing on the *embedded* resolutions of singularities which are minimal in the sense of [GL]. This can be thought as an embedded Nash correspondence.

The structure of the paper is as follows: the second section contains preliminaries on jet schemes. In the third section, we give and prove the main results. In the last section, we ask some questions about the Arc-Hilbert-Poincaré series [BMS1],[BMS2] for rational double point singularities, and about jet schemes of locally complete intersection rational singularities.

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2 Preliminaries on jet schemes

Let k be an algebraically closed field of arbitrary characteristic. Let X be a k-algebraic variety and let $m \in \mathbb{N}$. The functor $F_m : k-Schemes \longrightarrow Sets$ which to an affine scheme defined by a $k-$algebra A associates

$$F_m(Spec(A)) = Hom_k(SpecA[t]/(t^{m+1}), X)$$

is representable by a $k-$scheme X_m [V]. X_m is the m-th jet scheme of X, and F_m is isomorphic to its functor of points. In particular the closed points of X_m are in bijection with the $k[t]/(t^{m+1})$ points of X.
For $m, p \in \mathbb{N}, m > p$, the truncation homomorphism $A[t]/(t^{m+1}) \longrightarrow A[t]/(t^{p+1})$ induces a canonical projection $\pi_{m,p} : X_m \longrightarrow X_p$. These morphisms clearly verify $\pi_{m,p} \circ \pi_{q,m} = \pi_{q,p}$ for $p < m < q$, and they are affine mrophisms, so that they define a projective system whose limit is a scheme that we denote X_∞ and we call the arc space of X.
Note that $X_0 = X$. We denote the canonical projection $\pi_{m,0} : X_m \longrightarrow X_0$ by π_m, and by Ψ_m the canonial morphisms $X_m \longrightarrow X_0$.

3 Jet schemes of rational double point singularities

These are the locally complete intersection rational surface singularities. They are of five types. Embedded in $\mathbb{C}^3 = \text{Spec } \mathbb{C}[x, y, z]$, they are defined by the following equations:

$$A_n,\ n \in \mathbb{N} : xy - z^{n+1} = 0.$$

$$D_n,\ n \in \mathbb{N}, n \geq 4 : z^2 - x(y^2 + x^{n-2}) = 0.$$

$$E_6 : z^2 + y^3 + x^4 = 0\cdot$$

$$E_7 : x^2 + y^3 + yz^3 = 0.$$

$$E_8 : z^2 + y^3 + x^5 = 0\cdot$$

We will study the jet schemes of these types of singularities each apart.
if R is a ring, $I \subseteq R$ an ideal, we denote by $V(I)$ the subvariety of Spec R defined by I. Let $X = \text{Spec}A$, where $A = \frac{\mathbb{C}[x,y,z]}{f}$, for f an equation defining one of the above singularities. Let $m \in \mathbb{N}$ and let A_m be the ring of sections of X_m. For $g \in A_m$, we denote by $D^m(g) \subset X_m$ the open set defined by $D^m(g) := \text{Spec } A_{mg}$.

3.1 The A_n singularities

The A_n singularities are the toric surface singularities which are locally complete intersection. Their jet schemes have been studied in [Mo3]. We find the following:

Theorem 3.1. *For $m \in \mathbb{N}, n \geq 1$, The scheme of $m-th$ jets centered in the singular locus of an A_n Singularity is a locally complete intersection scheme. For $m \leq n$ this scheme has m irreducible components each of codimension $m+2$. For $m \geq n+1$, it has n irreducible components each of codimension $m+2$.*

3.2 The singularities $D_n, n \geq 4$

We look to the singularities D_{2n}, (the study of the case of D_{2n+1} is analaguous). Let $f(x,y,z) = z^2 - xy^2 - x^{2n-1} \in \mathbb{C}[x,y,z]$ and let $X \subset \mathbb{C}^3$ be the hypersurface defined by f. We write

$$f(\sum_{i=0}^{m} x_i t^i, \sum_{i=0}^{m} y_i t^i, \sum_{i=0}^{m} z_i t^i) = \sum_{i=0}^{i=m} F_i t^i \quad mod \quad t^{m+1}, \qquad (\diamond)$$

then the m-th jet scheme X_m of X is defined in

$$\mathbb{C}^{3(m+1)} = (\mathbb{C}^3)_m = \mathrm{Spec}\mathbb{C}[x_i, y_i, z_i, i = 0, \ldots, m]$$

by the ideal $I_m = (F_0, F_1, ..., F_m)$. Since the restriction of π_m to $\pi_m^{(-1)}(X \backslash 0)$ is a trivial fibration [I], we have that $\overline{\pi_m^{-1}(X \backslash 0)}$ is an irreducible component of X_m of codimension $m+1$ in $(\mathbb{C}^3)_m$, and we will prove below that the codimension of $\pi_m^{-1}(0)$ in $(\mathbb{C}^3)_m$ is $m+2$. This implies that X_m is irreducible for every $m \in \mathbb{N}$, since any irreducible component of X_m may have codimension at most $m+1$, being definded by $m+1$ equations. Note that the irreducibility of X_m, follows already from [Mu], but in this simple case we give a direct prove without an extra effort. From now on, all the codimensions of subvarieties of X_m are considered as codimensions in $(\mathbb{C}^3)_m$.
We now study the irreducible components of $X_m^0 = \pi_m^{-1}(0)$. Let I_m^0 be its defining ideal in $(\mathbb{C}^3)_m$. From the above expression of I_m, we have that $\sqrt{I_1^0} = (x_0, y_0, z_0)$ and $\sqrt{I_2^0} = (x_0, y_0, z_0, z_1)$ and so X_1^0 and X_2^0 are irreducible of codimensions 3 and 4 respectively.
We have that $\sqrt{I_3^0} = (x_0, y_0, z_0, z_1, x_1 y_1)$ what implies that X_3^0 has 2 irreducible components each of codimension 5.
We stratify X_3^0 as follows:

$$X_3^0 = (D^3(x_1) \cap X_3^0) \cup (D^3(y_1) \cap X_3^0) \cup (X_3^0 \cap V(x_1, y_1)). \qquad (1)$$

We sutdy respectively the variteties $\overline{\pi_{m,3}^{-1}(D^3(y_1) \cap X_3^0)}, \overline{\pi_{m,3}^{-1}(D^3(x_1) \cap X_3^0)}$ and $\pi_{m,3}^{-1}(X_3^0 \cap V(x_1, y_1))$.
For $m \geq 4$, the ring of sections of $\pi_{m,3}^{-1}(D^3(y_1) \cap X_3^0)$ is

$$\frac{\mathbb{C}[x_i, y_i, z_i, i = 0, \ldots, m]_{y_1}}{(x_0, y_0, z_0, z_1, x_1, F_4, \ldots, F_m)}. \qquad (2)$$

In this ring we have the congruences

$$F_n \equiv -y_1^2 x_{n-2} + G_n(x_2, \ldots, x_{n-3}, y_1, \ldots, y_{n-3}, z_2, \ldots, z_{n-2}). \qquad (3)$$

where $n \leq 2 \leq m$ and G_n are polynomials in the mentionned variables. Moreover, since we can divide by y_1^2, these equations are linear in x_{n-2} and permit to get rid of the variables $x_i, i = 2, \ldots, m-2$. We conclude that $\overline{\pi_{m,3}^{-1}(D^3(y_1) \cap X_3^0)}$ is irreducible of codimension $m+2$ (This linearity argument will be used repeatedly in this paper). We will show that $\overline{\pi_{m,3}^{-1}(D^3(y_1) \cap X_3^0)}$ is an irreducible component of X_m^0 for every $m \geq 4$, we call it $C_m^{Dy_1}$. Let us consider $H_m = (\pi_{m,4}^{-1}(D^4(x_1)) \cap V(x_0, y_0, z_0))_{red}$ where $m \geq 4$. Let J_m be its defining ideal in $(\mathbb{C}^3)_m \cap D^m(x_1)$. Seen in the open subscheme $D^4(x_1)$ of $(\mathbb{C}^3)_4$, we have that

$$H_4 = V(x_0, y_0, z_0, z_1, y_1, z_2),$$

and

$$H_5 = V(x_0, y_0, z_0, z_1, y_1, z_2, y_2).$$

This is because

$$F_4 \equiv z_2^2 \;\; mod \;\; (x_0, y_0, z_0, z_1, y_1)$$

and

$$F_5 \equiv x_1 y_2^2 \;\; mod \;\; (x_0, y_0, z_0, z_1, y_1, z_2).$$

For $2 \leq k \leq n-1$, we have that in the open subscheme $D^{2k}(x_1)$ of $(\mathbb{C}^3)_{2k}$

$$H_{2k} = V(x_0, y_0, z_0, z_1, y_1, y_2, ..., y_{k-1}, z_2, z_3, ..., z_k)$$

and for $2 \leq k \leq n-2$,

$$H_{2k+1} = V(x_0, y_0, z_0, z_1, y_1, y_2, ..., y_k, z_2, z_3, ..., z_k)$$

in the open subscheme $D(x_1) \cap (\mathbb{C}^3)_{2k+1}$. This is due to the fact that for such a k, we have that

$$F_{2k} \equiv z_k^2 \;\; mod \;\; J_{2k-1}$$

and

$$F_{2k+1} \equiv x_1 y_k^2 \;\; mod \;\; J_{2k}.$$

which means that for $4 \leq m < 2n-1, H_m$ is irreducible of codimension $m+2$. For $m = 2n-1$, we have that

$$F_m \equiv x_1 y_{n-1}^2 - x_1^{2n-1} = x_1(y_{n-1} - x_1^{n-1})(y_{n-1} + x_1^{n-1}) \;\; mod \;\; J_{2n-2},$$

therefore H_{2n-1} has 2 irreducible components, each of codimension $2n+1$. For $m \geq 2n$, by the same argument as in equations 2 and 3, we have that the ring of sections of $\pi_{m,2n-1}^{-1}(D^{2n-1}(x_1) \cap V(x_0, y_0, z_0))$ is isomorphic to a polynomial ring over

$$\frac{\mathbb{C}[x_1, y_{n-1}]_{x_1}}{(y_{n-1} - x_1^{n-1})(y_{n-1} + x_1^{n-1})}. \qquad (4)$$

We then have that H_m has 2 irreducible components each of codimension $m+2$. We call their closures in X_m^0 : $C_m^{(Dx_1)}-$ and $C_m^{(Dx_1)}+$. We will show that $C_m^{(Dx_1)}-$ and $C_m^{(Dx_1)}+$ are irreducible components of X_m^0, for $m \geq 2n$.

By the stratification 1, it remains to look at $\pi_{m,3}^{-1}(X_3^0 \cap V(x_1,y_1))$. We have that

$$\pi_{4,3}^{-1}(X_3^0 \cap V(x_1,y_1)) = V(x_0,y_0,z_0,z_1,y_1,x_1,z_2)$$

which is not an irreducible component of X_4^0 since it has codimension 3 in $V(x_0,y_0,z_0,z_1)\cap(\mathbb{C}^3)_4$ while X_4^0 is only defined by 2 equations herein.
We have that

$$\pi_{5,3}^{-1}(X_3^0 \cap V(x_1,y_1)) = V(x_0,y_0,z_0,z_1,y_1,x_1,z_2)$$

which is irreducible of codimension 7, so it is an irreducible component of X_5^0.

$$\pi_{6,3}^{-1}(X_3^0 \cap V(x_1,y_1)) = V(x_0,y_0,z_0,z_1,y_1,x_1,z_2,z_3^2-x_2y_2^2)$$

which is an irreducible component of X_6^0. Indeed, it is irreducible because $z_3^2-x_2y_2^2$ is an irreducible polynomial; it is not included in $C_6^{(Dx_1)}$ or $C_6^{(Dy_1)}$ because if it is the case, they will be equal being irreducible and having the same dimension, but they are not equal because $\pi_{6,3}^{-1}(X_3^0 \cap V(x_1,y_1))$ is included in $V(x_1,y_1)$.
In order to determine the irreducible components of $\pi_{m,6}^{-1}(V(x_0,y_0,z_0,z_1,y_1,x_1,z_2,z_3^2-x_2y_2^2))$, we define the following varieties. For $k=2,\ldots,2n-3$, and $m>2k+2$ we set

$$C_m^{Dy_k} := \overline{\pi_{m,2k+2}^{-1}(D^{2k+2}(y_k)\cap V(x_0,x_1,y_0,y_1,\ldots,y_{k-1},z_0,z_1,,\ldots z_k))}.$$

By the same argument as in equations 2 and 3, we have that $C_m^{Dy_k}$ is irreducible of codimension $m+2$.
For $l=4,\ldots,2n-2$, we claim that

$$\pi_{2l,6}^{-1}(V(x_0,y_0,z_0,z_1,y_1,x_1,z_2,z_3^2-x_2y_2^2)) = \bigcup_{k=2,\ldots,l-2} C_{2l}^{Dy_k}\bigcup V(x_0,x_1,y_0,\ldots,y_{l-2},z_0,\ldots,z_{l-1},z_l^2-x_2y_{l-1}^2) \quad (5)$$

and for $l=2n-1$,

$$\pi_{4n-2,6}^{-1}(V(x_0,y_0,z_0,z_1,y_1,x_1,z_2,z_3^2-x_2y_2^2)) = \bigcup_{k=2,\ldots,2n-3} C_{4n-2}^{Dy_k}\bigcup V(x_0,x_1,y_0,\ldots,y_{2n-3},z_0,\ldots,z_{2n-2},z_{2n-1}^2-x_2y_{2n-2}^2-x_2^{2n-1}) \quad (6)$$

are the decompositions into irreducible components.
The proof of the claim is by induction on l. For $l=4$, we stratify $V(x_0,y_0,z_0,z_1,y_1,x_1,z_2,z_3^2-x_2y_2^2)$ as follows:

$$(V(x_0,y_0,z_0,z_1,y_1,x_1,z_2,z_3^2-x_2y_2^2)\cap V(y_2))\bigcup$$

$$(V(x_0, y_0, z_0, z_1, y_1, x_1, z_2, z_3^2 - x_2y_2^2) \cap D(y_2)). \quad (7)$$

Since we have that

$$\pi_{8,6}^{-1}(V(x_0, y_0, z_0, z_1, y_1, x_1, z_2, z_3^2 - x_2y_2^2) \cap V(y_2)) =$$

$$V(x_0, y_0, z_0, z_1, y_1, x_1, z_2, z_3, y_2, z_4^2 - x_2y_3^2),$$

and $\overline{\pi_{m,6}^{-1}(V(x_0, y_0, z_0, z_1, y_1, x_1, z_2, z_3^2 - x_2y_2^2) \cap D^6(y_2))}$ is $C_8^{Dy_2}$, we obtain the decomposition in the equality 5. This is the decomposition into irreducible components, because both parts have the same dimensions and are irredubile, so an inclusion of one part in the other would mean that they are equal, which is not true by their definitions.
If we suppose the claim is true for l, we deduce it for $l+1$ by stratifying

$$V(x_0, x_1, y_0, \ldots, y_{l-2}, z_0, \ldots, z_{l-1}, z_l^2 - x_2y_{l-1}^2)$$

as in the stratification (7).
We define

$$C^{4n-2}o := V(x_0, y_0, z_0, z_1, y_1, x_1, z_2, ..., z_{2n-2}, y_2, ..., y_{2n-3}, z_{2n-1}^2 - x_2y_{2n-2}^2 - x_2^{2n-1})$$

whcih is irreducible of codimension $4n$. For $m > 4n-2$, set

$$C^m o = \pi_{m,4n-2}^{-1}(C^{4n-2}o).$$

So we have defined the following irreducible subvarieties of X_{4n-2}^0:
$C_{4n-2}^{Dy_1}, C_{4n-2}^{Dy_2}, ..., C_{4n-2}^{Dy_{2n-3}}, C^{4n-2}o, C_{4n-2}^{Dx_1}-$ and $C_{4n-2}^{Dx_1}+$ each of codimension $m+2$.

These irreducible varieties are the irreducible components of X_{4n-2}^0 because by their definitions, their union is equal to X_{4n-2}^0, they are not equal. On the other hand, one of them cannot be contained in an other strictly because they have the same dimensions. We also recover the irreducible components of X_m^0 for every $m \leq 4n-2$ and we find that the codimension of each component is $m+2$. These components are not irreducible components of $X_m, m \leq 4n-2$ as explained in the beginning of this section, because the codimension of each is strictly bigger then $m+1$. Thus we have $\overline{\pi_m^{-1}(X \backslash 0)} = X_m$ and X_m is irreducible for $m \leq 4n-2$.
For $m > 4n-2$ we have by the construction of $C_m^{Dy_1}, C_m^{Dy_2}, ..., C_m^{Dy_{2n-3}}, C^m o, C_m^{Dx_1}-$ and $C_m^{Dx_1}+$, their union is equal to X_m^0. To prove that these are its irreducible components, we will prove that for $m > 4n-2, C^m o$ is irreducible of codimension $m+2$. Let $d := 4n-2$. For $m \geq d+1$ we have that $C^m o := \pi_{m,d}^{-1}(V(x_0, y_0, z_0, z_1, y_1, x_1, z_2, ..., z_{2n-2}, y_2, ..., y_{2n-3}, z_{2n-1}^2 - x_2y_{2n-2}^2 - x_2^{2n-1}))$ is defined in $(\mathbb{C}^3)_m$ by the ideal

$$(x_0, y_0, z_0, z_1, y_1, x_1, z_2, ..., z_{2n-2}, y_2, ..., y_{2n-3}, J_{m-d})$$

where J_{m-d} is the ideal obtained from the ideal defining X_{m-d} in $\mathbb{C}_{m-d}^3$ by changing variables. Indeed if we set

$$f\left(\sum_{i=2}^{m} x_i t^i, \sum_{i=n}^{m} y_i t^i, \sum_{i=2n-1}^{m} z_i t^i\right) =$$

$$f\left(t^2(\sum_{i=0}^{m-2} x_{2+i}t^i), t^{2n-2}(\sum_{i=0}^{m-(2n-2)} y_{2n-2+i}t^i), t^{2n-1}(\sum_{i=0}^{m-(2n-1)} z_{2n-1+i}t^i)\right) =$$

$$t^d f\left(\sum_{i=0}^{m-2} x_{2+i}t^i, \sum_{i=0}^{m-((2n-2))} y_{(2n-2)+i}t^i, \sum_{i=0}^{m-(2n-1)} z_{2n-1+i}t^i\right)$$

(the last equality follows from the fact that f is weighted homogenuous of degree d for the weights $2, 2n-2$ and $2n-1$ given respectively to x, y and z)

$$= t^d\left(\sum_{i=0}^{i=m-d} G_i t^i\right) \mod t^{m+1}, \qquad (\diamond\diamond)$$

then J_{m-d} is generated by $G_i, i = 0, \ldots, m-d$, and by comparing $(\diamond)$ with $(\diamond\diamond)$, we get that

$$G_i = F_i(x_2, \ldots, x_{2+i}, y_{2n-2}, \ldots, y_{2n-2+i}, z_{2n-1}, \ldots, z_{2n-1+i}).$$

We deduce that

$$\text{codim } (C^m o) = d + 1 + \text{codim } (X_{m-d}).$$

This implies by a simple induction on m that

$$\text{codim } (C^m o) = m + 2.$$

Therefore codim $(X_m^0) = m+2$, so X_m is irreducible. It follows that $C^m o$ which is isomorphic to a product of X_{m-d} by an affine space is irreducible. On the other hand, the ideal defining X_m^0 in $\mathbb{C}_m^3$ is generated by the $m+2$ functions $x_0, y_0, z_0, F_i, i = 2, \ldots, m$. Hence it is a complete intersection (see proposition 3.7 in [BMS1] for a more elegant proof of this fact). We deduce the following:

Theorem 3.2. *The scheme of $m-th$ jets centered in the singular locus of a D_n Singularity is a complete intersection scheme, and for $m \geq 2n-3$, the number of irreducible components of X_m^0 is equal to the number of exceptional curves on the minimal resolution of the singularity.*

Remark 3.3. *Pay attention. The shift in minoring m between the theorem and what comes before, is due to the fact that we were studying D_{2n} singularities, but the theorem is stated for D_n.*

3.3 The singularity E_6

Let $f(x, y, z) = z^2 + y^3 + x^4 \in \mathbb{C}[x, y, z]$ and let $X \subset \mathbb{C}^3$ be the variety defined by f. If we write

$$f(\sum_{i=0}^{m} x_i t^i, \sum_{i=0}^{m} y_i t^i, \sum_{i=0}^{m} z_i t^i) = \sum_{i=0}^{i=m} F_i t^i \quad mod \ \ t^{m+1},$$

then its m-th jet scheme X_m is defined in $\mathbb{C}^{3(m+1)} = (\mathbb{C}^3)_m$ by the ideal $I_m = (F_0, F_1, ..., F_m)$. As for D_n singularities, since the restriction of π_m to $\pi_m^{(-1)}(X \backslash 0)$ is a trivial fibration [I], we have that $\overline{\pi_m^{-1}(X\backslash 0)}$ is an irreducible component of X_m of codimension $m+1$ in $(\mathbb{C}^3)_m$, and we will prove below that the codimension of $\pi_m^{-1}(0)$ in $(\mathbb{C}^3)_m$ is $m+2$. This implies that X_m is irreducible for every $m \in \mathbb{N}$, since any irreducible component of X_m may have codimension at most $m+1$, being definded by $m+1$ equations. Note that the irreducibility of X_m, follows already from [Mu], but in this simple case we give a direct proof whithout an extra effort. From now on, all the codimensions of subvarieties of X_m are considered as codimensions in $(\mathbb{C}^3)_m$.
We now study the irreducible components of $X_m^0 = \pi_m^{-1}(0)$ defined by I_m^0 in $(\mathbb{C}^3)_m$. By the above expression of I_m, we have that $\sqrt{I_1^0} = (x_0, y_0, z_0)$, $\sqrt{I_2^0} = (x_0, y_0, z_0, z_1)$ and $\sqrt{I_3^0} = (x_0, y_0, z_0, z_1, y_1)$ which means that X_1^0, X_2^0 and X_3^0 are irreducible. $\sqrt{I_4^0} = (\sqrt{I_3^0}, z_2^2 + x_1^4)$ so that X_4^0 has 2 irreducible components, each of codimension 6.

We stratify X_0^4 as follows:

$$X_4^0 = (X_4^0 \cap D^4(z_2)) \cup (X_4^0 \cap V(z_2)) \tag{8}$$

For $m \geq 5$, we claim that $\overline{\pi_{m,4}^{-1}(D^4(z_2))}$ has 2 irreducible components, each of codimension $m+2$ and that we will call $C_m^{Dz_2}-$ and $C_m^{Dz_2}+$. The argument is the same as in equations (2) and (3) in the case of D_n singularities. Indeed, in the ring of sections of $\pi_{m,4}^{-1}(D^4(z_2))$

$$\frac{1}{2z_2} F^{(l)} = z_{l-2} - H^{(l)}, \text{with} \quad H^{(l)} \in k[z_2, \ldots, z_{l-3}, x_1, \ldots, x_l, y_2, \ldots, y_l]_{z_2}, \tag{9}$$

and the claim follows from the linearity of this equation in the $z_i, i \geq 3$. These components will be irreducible components of X_m^0 for every $m \geq 5$.
From the stratification (8), it remains to consider $\pi_{m,4}^{-1}(V(z_2)) = V(x_0, y_0, z_0, z_1, y_1, z_2, x_1) \cap X_m^0$ where $m \geq 5$. We have that

$$V(x_0, y_0, z_0, z_1, y_1, z_2, x_1) \cap X_5^0 = V(x_0, y_0, z_0, z_1, y_1, z_2, x_1)$$

is irreducible and of codimension 7 what means that it is an irreducible component of X_5^0 that we call B_5. X_5^0 has then 3 irreducible components each of codimension 7.(Note that these irreducible varieties are the irreducible components because by their definitions they are not equal, and one of them cannot be contained in an other strictly because of their dimensions).
We have that

$$V(x_0, y_0, z_0, z_1, y_1, z_2, x_1) \cap X_6^0 = V(x_0, y_0, z_0, z_1, y_1, z_2, x_1, z_3^2 + y_2^3)$$

which is irreducible of codimension 8, and X_6^0 has 3 irreducible components each of codimension 8. We stratify $A := V(x_0, y_0, z_0, z_1, y_1, z_2, x_1, z_3^2 + y_2^3)$ as follows

$$A = (A \cap D^6(z_3)) \cup (A \cap V(z_3)).$$

For $m \geq 7$, we have that $\overline{\pi^{-1}_{m,6}(A \cap D^6(z_3))}$ is irreducible of codimension $m+2$. We call this component $C_m^{Dz_3}$, it will be an irreducible component of X_m^0 for every $m \geq 7$.
Let us study

$$\pi^{-1}_{m,6}(A \cap V(z_3)) = V(x_0, y_0, z_0, z_1, y_1, z_2, x_1, z_3, y_2) \cap X_m^0$$

where $m \geq 7$. We have that

$$V(x_0, y_0, z_0, z_1, y_1, z_2, x_1, z_3, y_2) \cap X_7^0 = V(x_0, y_0, z_0, z_1, y_1, z_2, x_1, z_3, y_2),$$

so it is irreducible of codimension 9 and it gives rise to an irreducible component of X_7^0 that we call B_7. Then, X_7^0 has 4 irreducible components which are $C_m^{Dz_2}-, C_m^{Dz_2}+, C_m^{Dz_3}$ and $V(x_0, y_0, z_0, z_1, y_1, z_2, x1, z_3, y_2)$, each of codimension 9.
We have that $B := V(x_0, y_0, z_0, z_1, y_1, z_2, x_1, z_3, y_2) \cap X_8^0 =$

$$V(x_0, y_0, z_0, z_1, y_1, z_2, x1, z_3, y_2, (z_4 + ix_2)(z_4 - ix_2))$$

has 2 irreducible components each of codimension 10, therefore X_8^0 has 5 irreducible components each of codimension 10.

We stratify B as follows

$$B = (B \cap D^8(z_4)) \cup (B \cap V(z_4)). \tag{10}$$

We have that, for the same argument as in (9), $\overline{\pi^{-1}_{m,4}(B \cap D^4(z_2))}$ has 2 irreducible components, each of codimension $m+2$ and we will call them $C_m^{Dz_4}-$ and $C_m^{Dz_4}+$. These components will be irreducible components of X_m^0 for every $m \geq 9$.
From the stratification 10, it remains to consider

$$\pi^{-1}_{m,8}(B \cap V(z_4)) = V(x_0, y_0, z_0, z_1, y_1, z_2, x_1, z_3, y_2, z_4, x_2) \cap X_m^0,$$

where $m \geq 9$. We have that

$$V(x_0, y_0, z_0, z_1, y_1, z_2, x_1, z_3, y_2, z_4, x_2) \cap X_9^0 = V(x_0, y_0, z_0, z_1, y_1, z_2, x_1, z_3, y_2, z_4, x_2, y_3)$$

which is irreducible of codimension 12, and embedded in $V(x_0, y_0, z_0, z_1, y_1, z_2, x_1, z_3, y_2)$ it is of codimension 3, which means that it cannott be an irreducible components of

$$V(x_0, y_0, z_0, z_1, y_1, z_2, x_1, z_3, y_2) \cap X_9^0$$

which is defined in

$$V(x_0, y_0, z_0, z_1, y_1, z_2, x_1, z_3, y_2)$$

by the 2 equations

$$z_4^2 + x_2^4 = 2z_4z_5 + 4x_2^3x_3 + y_3^3 = 0.$$

Therefore X_9^0 has just 5 irreducible components only, each of codimension 11.
For the same reason and since

$$V(x_0, y_0, z_0, z_1, y_1, z_2, x_1, z_3, y_2, z_4, x_2) \cap X_{10}^0 = V(x_0, y_0, z_0, z_1, y_1, z_2, x_1, z_3, y_2, z_4, x_2, y_3, z_5)$$

has codimension 13 it cannot be an irreducible component and we have that X^0_{10} has again only 5 irreducible components, each of codimension 12.

$$V(x_0, y_0, z_0, z_1, y_1, z_2, x_1, z_3, y_2, z_4, x_2) \cap X^0_{11} = V(x_0, y_0, z_0, z_1, y_1, z_2, x_1, z_3, y_2, z_4, x_2, y_3, z_5)$$

is of codimension 13 which is equal to the codimensions of the other 5 components, so it is a new component of X^0_{11} that has born, and that we will call $C^{11}o$, so that X^0_{11} has 6 irreducible components.
We have that

$$X^0_{12} \cap V(x_0, y_0, z_0, z_1, y_1, z_2, x_1, z_3, y_2, z_4, x_2, y_3, z_5) =$$
$$V(x_0, y_0, z_0, z_1, y_1, z_2, x_1, z_3, y_2, z_4, x_2, y_3, z_5, z_6^2 + y_4^3 + x_3^4)$$

which is irreducible of codimension 14 and X^0_{12} has 6 irreducible components each of codimension 14. So we have shown that X_m is irreducible for $m \leq 12$. On the other hand, because f is weighted-homogeneous, we remark that the equations defining

$$V(x_0, y_0, z_0, z_1, y_1, z_2, x_1, z_3, y_2, z_4, x_2, y_3, z_5) \cap X_m$$

for $m \geq 12$ are the same defining X_{m-12} but in other variables (see the case of D_n singularities for a proof), which proves that X_m is irreducible for every m and therefore

$$C^m o := X^0_m \cap V(x_0, y_0, z_0, z_1, y_1, z_2, x_1, z_3, y_2, z_4, x_2, y_3, z_5).$$

is irreducible of codimension $m+2$ for every $m \geq 11$ and X^0_m has 6 irreducible components for every $m \geq 11$.
We deduce the following theorem

Theorem 3.4. *The scheme of $m-th$ jets centered in the singular locus of an E_6 Singularity is a complete intersection scheme, and for* $m \geq 11$, *the number of irreducible components of X^0_m is equal to the number of exceptional curves on the minimal resolution of the singularity.*

We also obtain the following infinite projective systems of irreducible components, induced by the restriction of the morphisms $\pi_{m,m-1}$:

$$\ldots \longrightarrow C_m^{Dz_2}- \longrightarrow \ldots \longrightarrow C_5^{Dz_2}- \longrightarrow C_4^{Dz_2}- \longrightarrow X_3^0, \tag{11}$$

$$\ldots \longrightarrow C_m^{Dz_2}+ \longrightarrow \ldots \longrightarrow C_5^{Dz_2}+ \longrightarrow C_4^{Dz_2}+ \longrightarrow X_3^0, \tag{12}$$

$$\ldots \longrightarrow C_m^{Dz_3} \longrightarrow \ldots \longrightarrow C_6^{Dz_3} \longrightarrow B_5 \longrightarrow C_4^{Dz_2}\pm, \tag{13}$$

$$\ldots \longrightarrow C_m^{Dz_4}- \longrightarrow \ldots \longrightarrow C_8^{Dz_4}- \longrightarrow B_7 \longrightarrow C_6^{Dz_3}, \tag{14}$$

$$\ldots \longrightarrow C_m^{Dz_4}+ \longrightarrow \ldots \longrightarrow C_8^{Dz_4}+ \longrightarrow B_7 \longrightarrow C_6^{Dz_3}, \tag{15}$$

$$\ldots \longrightarrow C^m o \longrightarrow \ldots \longrightarrow C^{11}o \longrightarrow C_{10}^{Dz_4} \pm . \tag{16}$$

We now will associate with an irreducible component of X^0_m a divisorial valuation over $\mathbb{C}^3$. For $m \in \mathbb{N}$, let $\psi^a_m : \mathbb{C}^3_\infty \longrightarrow \mathbb{C}^3_m$ be the canonical morphism, here the exponent a stands for ambiant.

For $p \in \mathbb{N}$, we consider the following cylinder in the arc space

$$Cont^p(f) = \{\gamma \in \mathbb{C}^3_\infty; \mathrm{ord}_t f \circ \gamma = p\}.$$

Since ψ^a_m is a trivial fibration, for every irreducible component $H_m \subset X^0_m$, we have that

$$\psi^{a\ -1}_m(H_m) \cap Cont^{m+1}(f)$$

is an irreducible component of $Cont^{m+1}(f)$. Note that fact that for every irreducible component $H_{m+1} \subset X^0_{m+1}$, such that $\pi_{m+1,m}(H_{m+1}) \subset H_m$, we have that $\mathrm{codim}(H_{m+1}) > \mathrm{codim}(H_m)$, implies that $\psi^{a\ -1}_m(H_m) \cap\ Cont^{m+1}(f) \neq \emptyset$. We associate to H_m a discrete valuation ν_{H_m} as follows: let γ be the generic point of $\psi^{a\ -1}_m(H_m) \cap\ Cont^{m+1}(f)$, then for every $h \in \mathbb{C}[x,y,z]$, we set

$$\nu_{H_m}(h) = \mathrm{ord}_t h \circ \gamma.$$

It follows from corollary 2.6 in [ELM], that ν_{H_m} is a divisorial valuation (see also [dFEI], [Re3], prop. 3.7 (vii) applied to $\psi^{a\ -1}_m(H_m)$).

Given $m \geq 1$, with an irreducible component H_m of X^0_m, we associate the following vector:

$$v(H_m) = (\nu_{H_m}(x), \nu_{H_m}(y), \nu_{H_m}(z)) \in \mathbb{N}^3.$$

We define the following set of divisorial valuations on $\mathbb{C}^3$:

$$EE := \{\nu_{H_m}; H_m \subset X^0_m, m \geq 1 \text{ is an irreducible component and}$$

$$v(H_m) \neq v(H_{m-1}) \text{ for } H_{m-1} \text{ a component verifying } \pi_{m,m-1}(H_m) \subset H_{m-1}\} \qquad (17)$$

Theorem 3.5. *The elements of EE are the divisorial valuations which appear on the minimal embedded resolutions of singularities of E_6.*

proof : From a direct analysis of the irreducible components in the projective systems (11),…,(16), we conclude that

$$EE = \{X^0_1, X^0_2, X^0_3, B_5, B_7, C^{11}o\}.$$

Moreover, the elements of EE are irreducible components which are defined in $\mathbb{C}^3_m$ by hyperplane coordinates. This implies that for every $H \in EE$, ν_H is a monomial valuation, which is defined by the vector $v(H_i) = a = (a_1, a_2, a_3)$, i.e. if $h = \sum_{i \in \mathbb{N}^3} a_i x^{i_1} y^{i_2} z^{i_3} \in \mathbb{C}[x,y,z]$ then

$$\nu_{H_i}(h) = \min_{i \in \mathbb{N}^3; b_i \neq 0}\ a_1 i_1 + a_2 i_2 + a_3 i_3.$$

We have these vectors: $v(X^0_1) = (1,1,1)$, $v(X^0_2) = (1,1,2)$, $v(X^0_3) = (1,2,2)$, $v(B_5) = (2,2,3)$, $v(B_7) = (2,3,4)$, $v(C^{11}o) = (3,4,6)$.

On the other hand, it follows from [GL](page 8,9,10) that there are five minimal embedded resolutions of singularities of E_6. These resolutions are minimal in the sense that if we contract one of the minimal divisors, we loose the smoothness of the srtict transform or of the ambiant space , or the normal crossings. We also have from [GL], page 9, that the divisorial valuations which are defined by the exceptional divisors appearing on these resolutions of singularities, are monomial valuations, and they are defined exactly by the vectors $v(H), H \in EE$ (modulo a permutation which is due to the fact that the authors of [GL] write the equation of E_6 as follows $x^2 + y^3 + z^4 = 0$). This terminates the proof. □

Remark 3.6. *In this simple case, the definition of EE is affected by the fact that E_6 is a singularity which is non-degenerate with respect to its Newton Polygon. In general, this definition needs careful study of the divisorial valuations defined by the irreducible components of the jet schemes [LMR],[T]. Theorem 3.5 should be thought as an embedded Nash correspondence [ELM],[LMR].*

We get a graph by representing each irreducible component of $X_m^0, m \geq 1$, by a vertex $v_{i,m}, 1 \leq i \leq N(m)(N(m)$ is the number of irreducible components of $X_m^0)$ and by joining the vertices $v_{i_1,m+1}$ and $v_{i_0,m}$ if $\pi_{m+1,m}$ induces one of the maps appearing in the projective systems (11),...,(12) between the corresponding irreducible components.We represent this graph in figure 1. The surrounded vertices are the vertices which represent elements of EE. We remark that for m bigger than 11, the number of vertices counted horizontally is 6.

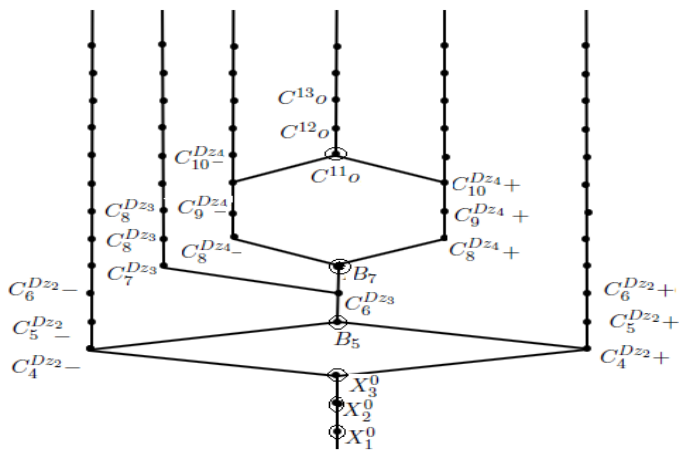

Figure 1

4 The singularities E_7 and E_8

The same arguments that we have used to study the jet schemes of the singularities A_n, D_n and E_6 work also for the jet schemes of the singularities E_7 and E_8 and we found:

Theorem 4.1. *The scheme of $m-th$ jets centered in the singular locus of the Singularities E_7 (resp. E_8) is a complete intersection scheme, and for $m \geq 17$(resp. 29) the number of irreducible components of X_m^0 is equal to the number of exceptional curves on the minimal resolution of the singularity.*

4.1 Questions

In this section, we ask some questions related to jet schemes of rational singularities.

Let $X = Spec\ \frac{k[x_0,...,x_n]}{(f_1,...,f_r)}$ be an affine $k-$scheme, where k is a field. We assume that the point O defined by the ideal $(x_1, ..., x_n)$ belongs to X. The rings of globlal sections of $\Gamma(X_m)$ and $\Gamma(X_\infty)$ are graded rings (see [BMS1] or [BMS2] for details). The ring of sections $B := \Gamma(X_\infty^0)$ of the fiber above the point «O» of $\Psi_m : X_\infty \longrightarrow X$ is also graded, and we can associate to it the Arc Hilbert-Poincaré series:

$$AHP_{X,O}(t) = \sum_{m \in \mathbb{N}} rg_k(B_m)t^m$$

where B_m is the homogenuous component of B of degree m and $rk_k(B_m)$ is its rank over k as a $k-$vector space.

Remark 4.2. *Note that for $m \geq 1$, the ring of sections $B := \Gamma(X_m^0)$ of X_m^0 is also graded. We denote by $P_{X,O}^{(m)}(t)$ its poincaré series. By the definition of the grading, we have that for every $m \geq 1$, $AHP_{X,O}(t) = P_{X,O}^{(m)}(t) \mod t^m$.*

We will use the following theorem to compute the Arc-Hilbert Poincaré series for rational double point singularities.

Theorem 4.3. *[S] Let R be a k graded algebra. Let $\theta_1, ..., \theta_r$ be a regular sequence of nonzero homogeneous elements of R of positive degree, say deg $\theta_i = d_i$. Let I be the ideal generated by the θ_i and S the quotient of R by I endowed with the natural "quotient grading". then*

$$P(S,t) = \sum_{m \in \mathbb{N}} rg_k(S_m)t^m = P(R,t) \prod_{i=1}^{r} (1 - t^{d_i}).$$

By theorems 3.1,3.2,3.4 an 4.1 we have that the jet schemes centered at a rational double point singularity are complete intersection. We can then apply theorem 4.3 and then remark 4.2 to obtain:

Proposition 4.4. *If X is a surface with a rational double point singularity at O, then:*

$$AHP_{X,0}(t) = \frac{1}{(1-t)^3} \frac{1}{(1-t^2)^2 ... (1-t^m)^2 ...}$$

We can find a more general statement than proposition 4.4 in [BMS1].

Question 4.5. *Does the Arc Hilbert-Poincaré series characterize rational double point singularities ?*

Let X be a singular locally complete intersection variety over $\mathbb{C}-$algebraic variety and let $sing(X)$ be its singular locus. We denote by $X_m^{sing} := \pi_m^{-1}(sing(X))$.

Question 4.6.

- *If X has at most rational singularities, is the number of irreducible components of X_m^{sing} independent of m, for m big enough ?*
- *Suppose that the number of irreducible components of X_m^{sing} is independent of m, for m big enough, does X have at most rational singularities ?*

We think that the answer to the questions in 4.6 is yes.

Question 4.7. *If the answer to the first question in 4.6 is yes, is the number of irreducible components of X_m^{sing}, for m big enough, equal to the number of irreducible components of the space of arcs centered in the singular locus of X.*

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